The Exponential Distribution

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The exponential distribution was first used in the study of arrival times. An arrival time is the length of time that we have to wait before the realization of an event. An important property of the exponential distribution is memorylessness, which is knowing that the event did not occur during the time interval $[0,\, t]$ does not help us in predicting if the event will occur during the time interval $[t,\, t+\Delta t]$. In other words the future is independent of the past. Events that are generally thought to be exponentially distributed are...

Table 1: Exponentially Distributed Events

Natural disasters and losses due to natural disasters
Equipment failure rates
The number of calls that a call center receives over a given time interval
Number of accidents at an intersection over a given time interval
Corporate bond defaults
Arrival of news that causes stock prices to jump

An event such as death is not exponentially distributed because the memorylessness property is violated. As an example, given that a person has lived 70 years does help us in predicting if that person will die over the next 10 years. The probability of death is possitively correlated with age, which is obvious.

A Hypothetical Problem

We are tasked with valuing a pizzeria at the base of a volcano. If and when the volcano erupts the business will cease to exist. The expected eruption date is in 10 years but the volcano could erupt at any time.

Part I - We want to simulate arrival times of the eruption and we determine that the exponential distribution is the probability distribution from which we will pull random arrival times. The random number that we pull from a uniforn distribuion with domain zero to one is 0.82. What is the exponentially-distributed arrival time associated with this uniformly-distributed random number?

Part II - What is the probability that the volcano will erupt during year one? Given that it is now year 10 and the volcano has not erupted what is the probability that the volcano will erupt during year 11?

The Mathematics of the Exponential Distribution

The probability density function (PDF) applicable to the exponential distribution is...

$$f(u) = \lambda\, e^{-\lambda u} \tag{1}$$

The domain of the exponential distribution is the range $[0,\, \infty]$. To be a valid probability distribution the integral of the exponential distribution's PDF over the exponential distribution's domain must equate to one. The following integration proves that point...

$$
\begin{aligned}
\int_{u=0}^{u=\infty} f(u)\,\delta u &= \int_{u=0}^{u=\infty} \lambda\, e^{-\lambda u}\,\delta u \\
&= \lambda \int_{u=0}^{u=\infty} e^{-\lambda u}\,\delta u \\
&= \lambda \left\{ -\frac{1}{\lambda}\, e^{-\lambda u} \Big|_{u=0}^{u=\infty} \right\} \\
&= \lambda \left\{ -\frac{1}{\lambda} \Big(0 - 1 \Big) \right\} \\
&= 1
\end{aligned} \tag{2}
$$

Because...

$$
\lim_{u \to \infty} e^{-\lambda u} = 0 \ \ ...and...\ \ e^{-\lambda(0)} = 1 \tag{3}
$$

If the random variable Z is the arrival time of the event then the probability of the event occurring sometime before time t where $t > 0$ is...

$$
\begin{aligned}
Prob\Big[Z < t \Big] &= \int_{u=0}^{u=t} \lambda\, e^{-\lambda u}\,\delta u \\
&= \lambda \int_{u=0}^{u=t} e^{-\lambda u}\,\delta u \\
&= \lambda \left\{ -\frac{1}{\lambda}\, e^{-\lambda u} \Big|_{u=0}^{u=t} \right\} \\
&= \lambda \left\{ -\frac{1}{\lambda} \Big(e^{-\lambda t} - 1 \Big) \right\} \\
&= 1 - e^{-\lambda t}
\end{aligned} \tag{4}
$$

The probability that the arrival time of the event will occur sometime after time t where $t > 0$ is...

$$
\begin{aligned}
Prob\Big[Z > t \Big] &= \int_{u=t}^{u=\infty} \lambda\, e^{-\lambda u}\,\delta u \\
&= \lambda \int_{u=t}^{u=\infty} e^{-\lambda u}\,\delta u \\
&= \lambda \left\{ -\frac{1}{\lambda}\, e^{-\lambda u} \Big|_{u=t}^{u=\infty} \right\} \\
&= \lambda \left\{ -\frac{1}{\lambda} \Big(0 - e^{-\lambda t} \Big) \right\} \\
&= e^{-\lambda t}
\end{aligned} \tag{5}
$$

Because...

$$
\lim_{u \to \infty} e^{-\lambda u} = 0 \tag{6}
$$

The defining characteristic of the exponential distribution is that it is memoryless. Given the time interval $[0,\, s,\, t]$ where $0 < s < t$ the probability that the event will not occur during the time interval $[s,\, t]$ given that the event has not occurred during the time interval $[0,\, s]$ is...

$$
\begin{aligned}
P[Z > t - s \,|\, Z > s] &= \frac{P[Z > s] \cup P[Z > t - s]}{P[Z > s]} \\
&= \frac{P[Z > t]}{P[Z > s]} \\
&= \frac{e^{-\lambda t}}{e^{-\lambda s}} \\
&= e^{-\lambda t}\, e^{\lambda s} \\
&= e^{-\lambda(t-s)}
\end{aligned} \tag{7}
$$

Note that the probability of the event not occurring during the time interval $[s,\, t]$ is the exponential of λ times the length of the time interval $[s,\, t]$. The length of the time interval $[0,\, s]$ over which the event did not occur is irrelevant.

Often times we want to pull random arrival times from the exponential distribution. The first step in deriving the equation for the random arrival time is to use Equation (4) above, which defines the probability that the event will occur sometime before time t, and solve for t. This equation is...

$$\begin{aligned} Prob\Big[Z < t\Big] &= 1 - e^{-\lambda t} \\ e^{-\lambda t} &= 1 - Prob\Big[Z < t\Big] \\ -\lambda t &= \ln\left\{1 - Prob\Big[Z < t\Big]\right\} \\ t &= -\ln\left\{1 - Prob\Big[Z < t\Big]\right\} \div \lambda \end{aligned} \tag{8}$$

The next step is to replace $Prob[Z < t]$ in the equation above with a random number pulled from a uniform distribution. If we define $\hat{t}$ to be the random arrival time and θ to be a random number pulled from a uniform distribution then the equation for simulating an exponentially distributed arrival time is...

$$\hat{t} = \frac{-\ln(1-\theta)}{\lambda} \tag{9}$$

The Mean and Variance of the Exponential Distribution

The first moment of the exponential distribution is the expected value of the exponentially-distributed random variable Z, which is...

$$\begin{aligned} \mathbb{E}\Big[Z\Big] &= \int_{u=0}^{u=\infty} u\, f(u)\, \delta u \\ &= \int_{u=0}^{u=\infty} u\, \lambda\, e^{-\lambda u}\, \delta u \\ &= -\frac{1+\lambda u}{\lambda}\, e^{-\lambda u}\Bigg|_{u=0}^{u=\infty} \\ &= 0 - \left(-\frac{1}{\lambda}\right) \\ &= \frac{1}{\lambda} \end{aligned} \tag{10}$$

The second moment of the exponential distribution is the expected value of the square of the exponentially-distributed random variable Z, which is...

$$\begin{aligned} \mathbb{E}\Big[Z^2\Big] &= \int_{u=0}^{u=\infty} u^2 f(u)\, \delta u \\ &= -\frac{\lambda^2 u^2 + 2\lambda u + 2}{\lambda^2}\, e^{-\lambda u}\Bigg|_{u=0}^{u=\infty} \\ &= 0 - \left(-\frac{2}{\lambda^2}\right) \\ &= \frac{2}{\lambda^2} \end{aligned} \tag{11}$$

The mean of the exponential distribution is...

$$\begin{aligned} mean &= \mathbb{E}\Big[Z\Big] \\ &= \frac{1}{\lambda} \end{aligned} \tag{12}$$

The variance of the exponential distribution is...

$$\begin{aligned} variance &= \mathbb{E}\Big[Z^2\Big] - \Big[\mathbb{E}\Big[Z\Big]\Big]^2 \\ &= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 \\ &= \frac{1}{\lambda^2} \end{aligned} \tag{13}$$

The Answer to our Hypothetical Problem

The exponential distribution is often times a convenient distribution to use because we only have to estimate one parameter (λ). The expected arrival time of the volcano's eruption is 10 years hence. Using Equation (12) our estimate of λ is...

$$\begin{aligned} mean &= \frac{1}{\lambda} \\ 10 &= \frac{1}{\lambda} \\ \lambda &= 0.10 \end{aligned} \tag{14}$$

Part I - Given that 0.82 was pulled from the uniform distribution and our estimate of λ in the equation above, using Equation (9) the random arrival time to use in our simulation is...

$$\begin{aligned} \hat{t} &= \frac{-\ln(1-\theta)}{\lambda} \\ &= \frac{-\ln(1-0.82)}{0.10} \\ &= 17.15 \;\; years \end{aligned} \tag{15}$$

Part II - Using Equation (4) the probability of the volcano erupting in year one is...

$$Prob\Big[Z < 1.00\Big] = 1 - e^{-0.10\times 1} = 0.0952 \tag{16}$$

Using Equation (7) the probability of the volcano erupting in year 11 given that it did not erupt prior to or including year 10 is...

$$P[Z > 11 - 10 \,|\, Z > 10] = e^{-\lambda(t-s)} = e^{-0.10\times(11-10)} = 0.0952 \tag{17}$$